THE TOPOLOGY OF THE SET OF NONSOLITON LIE ALGEBRAS IN THE MODULI SPACE OF NILPOTENT LIE ALGEBRAS

TRACY L. PAYNE

ABSTRACT. A Lie algebra is called *nonsoliton* if it does not admit a soliton inner product. We demonstrate that the subset of nonsoliton Lie algebras in the moduli space of indecomposable n-dimensional $\mathbb{N}$-graded nilpotent Lie algebras is discrete if and only if $n \leq 7$.

1. INTRODUCTION

A Lie algebra may be endowed with infinitely many different inner products. Among these, soliton inner products are considered preferred inner products. An inner product Q on a nilpotent Lie algebra $\mathfrak{g}$ is called *soliton* if the Ricci endomorphism Ric of $\mathfrak{g}$ defined by Q differs from a derivation of $\mathfrak{g}$ by a scalar multiple of the identity map on $\mathfrak{g}$. (See Section 2.2 for a precise definition of the map Ric). We will call a Lie algebra *soliton* if it admits a soliton inner product and we will call it *nonsoliton* if it does not admit a soliton inner product.

Soliton inner products on nilpotent Lie algebras are called *nilsoliton*. The study of nilsoliton inner products for nilpotent Lie algebras originated in the analysis of Einstein solvmanifolds ([Lau01]). Indeed, deep results of Heber and Lauret allow one to reduce the study of Einstein inner products on solvable Lie algebras to the study of soliton inner products on nilradicals ([Heb98], [Lau01], [Lau10]). Of independent interest, a soliton inner product on a nilpotent Lie algebra defines a metric on the corresponding simply connected nilpotent Lie group that is soliton in the sense that the Ricci flow moves the metric by diffeomorphisms and rescaling ([Lau01]). And, outside of the category of

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homogeneous spaces, soliton inner products are of use in a purely algebraic context in that they supply extra structure for algebraic computations and may give canonical presentations of Lie algebras (See Example 3.11 of [Pay11]).

When they exist, nilsoliton inner products are unique up to scaling ([Lau01]). If a nilpotent Lie algebra admits a soliton inner product, then it is $\mathbb{N}$-graded. As not all nilpotent Lie algebras are $\mathbb{N}$-graded, not all nilpotent Lie algebras admit nilsoliton inner products. One can find continuous families of nonsoliton nilpotent Lie algebras by finding continuous families of characteristically nilpotent Lie algebras. (A Lie algebra is *characteristically nilpotent* if its derivation algebra is nilpotent.) Such families exist in dimensions seven and higher (see [Kha02]). As the direct sum of nilpotent Lie algebras is soliton if and only if each summand is soliton ([Jab11], [Nik11]), we will restrict our attention to indecomposable nilpotent Lie algebras. We will study the subset of nonsoliton Lie algebras in the moduli space of all indecomposable $\mathbb{N}$-graded nilpotent Lie algebras of fixed dimension. In particular, we are interested in determining when the set of nonsoliton Lie algebras is discrete in this moduli space.

In dimensions 6 and lower, the situation is well-understood: the moduli space of nilpotent Lie algebras is discrete ([dG07]), and all nilpotent Lie algebras of dimension 6 and lower admit soliton inner products ([Lau02], [Wil03]). In dimension 7, the moduli space of real nilpotent Lie algebras consists of a finite number of discrete points and a finite number of continuous families of nonisomorphic nilpotent Lie algebras ([See93], [Gon98]). Nikolayevsky proved that if two real nilpotent Lie algebras have the same complexification, then either both are soliton or both are nonsoliton ([Nik11]). This result was also proved independently by M. Jablonski using different methods ([Jab]). Using this result about complex forms, along with Carles's classification of complex nilpotent Lie algebras of dimension 7 ([Car96], [Mag07]), Culma determined precisely which 7-dimensional complex nilpotent Lie algebras have real forms that admit nilsoliton inner products ([Cul11a], [Cul11b]). Culma found that among the continuous families of nonsoliton nilpotent Lie algebras in dimension 7, none of them are $\mathbb{N}$-graded. It follows that the subset of nonsoliton Lie algebras in the moduli space of indecomposable 7-dimensional $\mathbb{N}$-graded nilpotent Lie algebras is discrete.

Arroyo determined precisely which $\mathbb{N}$-graded filiform nilpotent Lie algebras of dimension 8 admit soliton inner products ([Arr11]). She found although there are continuous families of solitons in that moduli space, there are precisely four isolated nonsoliton Lie algebras.

Eberlein and Nikolayevsky showed independently that except for in two cases, soliton Lie algebras are dense in the moduli space of two-step nilpotent Lie algebras ([Ebe08], [Nik11]). (Note that all two-step nilpotent Lie algebras are $\mathbb{N}$-graded by a derivation that equals the identity on a complement to the center and that is twice the identity on the center.) Jablonski showed that the soliton Lie algebras are dense in the remaining two cases– when the nilpotent Lie algebra is of type $(2k+1, 2)$ or $(2k+1, \binom{2k+1}{2} - 2)$ ([Jab08]).

Theorem 1.1 ([Ebe08], [Nik11], [Jab08]). *The set of soliton Lie algebras is dense in the moduli space of two-step nilpotent Lie algebras.*

Given the classification results we have described, and the theorem just stated, one might wonder if the set of nonsoliton Lie algebras is always discrete in the moduli space of two-step n-dimensional nonsoliton $\mathbb{N}$-graded nilpotent Lie algebras. The answer is no. The first continuous families of nonsoliton $\mathbb{N}$-graded nilpotent Lie algebras were found by C. Will.

Theorem 1.2 ([Wil10]). *The moduli space of indecomposable* 9*-dimensional two-step nilpotent Lie algebras contains two one-parameter families of nonsoliton nilpotent Lie algebras.*

Jablonski defined a general method for constructing families of two-step nilpotent Lie algebras called concatenation. He used the concatenation construction to define continuous families of irreducible nonsoliton two-step nilpotent Lie algebras in infinitely many dimensions.

Theorem 1.3 ([Jab11]). *For* $n \geq 23$, *the moduli space of indecomposable* n*-dimensional two-step nilpotent Lie algebras contains a continuous family of nonsoliton Lie algebras.*

There are two key issues involved in the results of Jablonski and Will. First of all, nonsoliton nilpotent Lie algebras are quite rare, and two-step nilpotent Lie algebras are not classified in dimensions 10 and higher. Therefore, finding examples of nonsoliton nilpotent Lie algebras, or curves of them, requires a thorough understanding of the structure of nilpotent Lie algebras and how that structure relates to the existence of a soliton inner product. Second, in contrast to the semisimple case, there are few fine algebraic invariants that allow one to distinguish nonisomorphic nilpotent Lie algebras, so it is a significant task to show that the curves of nonsoliton nilpotent Lie algebras are mutually nonisomorphic. Will used the Pfaffian defined by Scheuneman in [Sch67] to distinguish the Lie algebras in her families, while Jablonski used geometric invariant theory.

Our main result is that the moduli space of indecomposable $\mathbb{N}$-graded n-dimensional nonsoliton nilpotent Lie algebras is not discrete if $n \geq 8$:

Theorem 1.4. *The moduli space of indecomposable n-dimensional nonsoliton $\mathbb{N}$-graded nilpotent Lie algebras contains a one-parameter family of nonsolitons if $n \geq 8$.*

As a corollary we can say exactly when the nonsoliton Lie algebras are isolated in the moduli space:

Corollary 1.5. *The set of nonsoliton Lie algebras is discrete in the moduli space of indecomposable n-dimensional nonsoliton $\mathbb{N}$-graded nilpotent Lie algebras if and only if $n \leq 7$.*

The families of nilpotent Lie algebras that we construct to prove the theorem are the first examples of continuous families of three-step nonsoliton nilpotent Lie algebras. It would be interesting to refine the result in Theorem 1.4 by specializing it to the two-step case, determining in which dimensions nonsolitons are discrete in the moduli space of two-step nilpotent Lie algebras.

This manuscript is organized as follows. In Section 2, we review necessary background material related to nilpotent Lie algebras, inner products on Lie algebras, soliton inner products, and Nikolayevsky (pre-Einstein) derivations of nilpotent Lie algebras. In Section 3, we present two continuous families of indecomposable $\mathbb{N}$-graded nilpotent Lie algebras, one in dimension eight and one in dimension nine, and we prove that the Lie algebras in the families are nonsoliton. We also describe the derivation algebras of the Lie algebras in the families. In Section 4, we use the 8- and 9-dimensional examples from Section 3 to construct continuous families of indecomposable $\mathbb{N}$-graded nilpotent Lie algebras in dimensions $n \geq 10$. We find Nikolayevsky derivations for these Lie algebras, and we prove that all of the Lie algebras in the families are nonsoliton. Last, we prove that for any dimension $n \geq 8$, the Lie algebras in the family are all mutually nonisomorphic. In Section 5, we combine our results from Sections 3 and 4 to prove the main result.

2. Preliminaries

2.1. **Lie algebras.** The descending central series of a Lie algebra $\mathfrak{g}$ is defined by $\mathfrak{g}^{(1)} = \mathfrak{g}$ and $\mathfrak{g}^{(j+1)} = [\mathfrak{g}, \mathfrak{g}^{(j)}]$ for $j > 1$. The Lie algebra $\mathfrak{g}$ is *nilpotent* if and only if there is an integer r so that $\mathfrak{g}^{(r)}$ is trivial. If r is the smallest integer so that $\mathfrak{g}^{(r+1)}$ is trivial, then $\mathfrak{g}$ is said to be *r-step* nilpotent. An *r-step* nilpotent Lie algebra is said to of *type* $(n_1, n_2, \ldots, n_r)$ if $\dim(\mathfrak{g}^{(j)}/\mathfrak{g}^{(j+1)}) = n_j$ for $j = 1, \ldots, r$.

Let $\mathrm{Der}(\mathfrak{g})$ be the derivation algebra of $\mathfrak{g}$. The algebra $\mathrm{Der}(\mathfrak{g})$ has Levi decomposition $\mathfrak{s} \oplus (\mathfrak{t}_s \oplus \mathfrak{t}_c) \oplus \mathfrak{n}$ where $\mathfrak{s}$ is the semisimple Levi factor and the solvable radical $\mathrm{rad}(\mathrm{Der}(\mathfrak{g})) = \mathfrak{t} \oplus \mathfrak{n}$ is the direct sum of its nilradical $\mathfrak{n}$ and a torus $\mathfrak{t}$. The torus further decomposes as the sum $\mathfrak{t} = \mathfrak{t}_s \oplus \mathfrak{t}_c$ of an $\mathbb{R}$-split torus $\mathfrak{t}_s$ and a compact torus $\mathfrak{t}_c$. The dimension of $\mathfrak{t}$ is called the *rank* of $\mathfrak{g}$, and the dimension of the $\mathbb{R}$-split torus $\mathfrak{t}_s$ is called the *real rank* of $\mathfrak{g}$.

A Lie algebra is *indecomposable* if it cannot be written as the direct sum of two nontrivial ideals.

2.2. **Metric Lie algebras and soliton inner products.** A *metric Lie algebra* $(\mathfrak{g}, Q)$ is a Lie algebra $\mathfrak{g}$ endowed with an inner product Q. Associated to each metric Lie algebra is a unique homogeneous space (G, g), where G is the connected Lie group whose Lie algebra is $\mathfrak{g}$, and g is the left invariant metric on G such that that the restriction of g to the tangent space $T_eG \cong \mathfrak{g}$ of G at the identity coincides with Q. The Ricci endomorphism Ric for the Riemannian manifold (G, g), when restricted to T_eG, is an endomorphism of $T_eG \cong \mathfrak{g}$. We call Ric_e the Ricci endomorphism for the metric Lie algebra $(\mathfrak{g}, Q)$ and we abuse notation to let Ric denote Ric_e .

Let $(\mathfrak{n}, Q)$ be a metric nilpotent Lie algebras with associated homogeneous space (N, g). Then Ricci form ric_e for (N, g) at the identity is the bilinear form on $T_eN \cong \mathfrak{n}$ given by

$$\mathsf{ric}(x, y) = -\frac{1}{2}\sum_{i=1}^{n} Q([x, x_i], [y, x_i]) + \frac{1}{4}\sum_{i,j=1}^{n} Q([x_i, x_j], x)Q([x_i, x_j], y),$$

for $x, y \in \mathfrak{n}$, and where $\{x_i\}_{i=1}^n$ is an orthonormal basis for $\mathfrak{n}$. Then Ricci endomorphism $\mathrm{Ric}_e = \mathrm{Ric}$ at the identity is the endomorphism of $T_eN \cong \mathfrak{n}$ given by the condition that $Q(\mathrm{Ric}(x), y) = \mathsf{ric}(x, y)$, for all $x, y \in \mathfrak{n}$.

Let $(\mathfrak{n}_1, Q_1)$ and $(\mathfrak{n}_2, Q_2)$ be metric nilpotent Lie algebras with associated homogeneous spaces (N_1, g_1) and (N_2, g_2) respectively. A map $\psi : \mathfrak{n}_1 \to \mathfrak{n}_2$ naturally induces the map $\overline{\psi} : N_1 \to N_2$, where $\overline{\psi} = \exp \circ \psi \circ \exp^{-1}$. E. Wilson proved that $\overline{\psi}$ is an isometry if and only if ψ is an isometric isomorphism ([Wil82]); i.e. ψ is an isomorphism and $Q_1(x, y) = Q_2(\psi(x), \psi(y))$ for all $x, y \in \mathfrak{n}_1$.

A metric Lie algebra $(\mathfrak{g}, Q)$ is called *soliton* if its Ricci endomorphism $\mathrm{Ric} \in \mathrm{End}(\mathfrak{g})$ differs from a scalar multiple of the identity map by a derivation; that is, there exists a $\beta \in \mathbb{R}$ called the *soliton constant* so that $\hat{D} = \mathrm{Ric} - \beta\,\mathrm{Id}$ is a derivation. In the case that the Lie algebra is nilpotent, we call the inner product a *nilsoliton* inner product, we call

β the *nilsoliton constant* and we call the derivation $\hat{D}$ the *nilsoliton derivation*.

Let $\mathfrak{g}$ be a nonabelian Lie algebra with basis $\mathcal{B}$. Let Λ index the set of nonzero structure constants α_{ij}^k relative to $\mathcal{B}$, modulo skew-symmetry:

$$\Lambda = \{(i,j,k) \,:\, \alpha_{ij}^k \neq 0, i < j\}.$$

To each triple $(i,j,k) \in \Lambda$, we associate the *root vector* $\boldsymbol{y}_{ij}^k = \boldsymbol{e}_i + \boldsymbol{e}_j - \boldsymbol{e}_k$, where $\{\boldsymbol{e}_i\}_{i=1}^m$ is the standard basis for $\mathbb{R}^m$. Let $\boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_m$ be an enumeration of the root vectors $\boldsymbol{y}_{ij}^k, (i,j,k) \in \Lambda$, using some fixed ordering of $\mathbb{N}^3$. Our convention is to order $\mathbb{N}^3$ so that for $i_1, j_1, k_1, i_2, j_2, k_2 \in \mathbb{N}$,

- $(i_1, j_1, k_1) < (i_2, j_2, k_2)$ if $k_1 < k_2$
- $(i_1, j_1, k_1) < (i_2, j_2, k_1)$ if $i_1 < i_2$
- $(i_1, j_1, k_1) < (i_1, j_2, k_1)$ if $j_1 < j_2$.

The *Gram matrix* for $\mathfrak{g}$ with respect to $\mathcal{B}$ is the matrix $U = (u_{ij})$ whose entries are the inner products of the root vectors: $u_{ij} = \boldsymbol{y}_i \cdot \boldsymbol{y}_j$. We will use the following theorem of Nikolayevsky to show that the nilpotent Lie algebras in our families do not admit soliton inner products.

Theorem 2.1 (Theorem 3, [Nik11])**.** *Let $\mathfrak{n}$ be a nonabelian nilpotent Lie algebra with basis $\mathcal{B}$. Suppose that the Gram matrix U for $\mathfrak{n}$ with respect to $\mathcal{B}$ has no entries of 2. Then $\mathfrak{n}$ admits a soliton inner product if and only if the matrix equation $U\boldsymbol{v} = [1]$ has a solution $\boldsymbol{v}$ with all positive entries.*

Note that we have replaced Nikolayevsky's hypothesis that the basis is a "nice basis" with an equivalent hypothesis on the Gram matrix U (which depends only on the index set Λ).

2.3. **Nikolayevsky derivations.** A derivation D^N of a Lie algebra $\mathfrak{g}$ is called a *Nikolayevsky derivation* if it is semisimple with real eigenvalues and

$$\operatorname{trace}(D^N \circ F) = \operatorname{trace}(F) \tag{1}$$

for all $F \in \operatorname{Der}(\mathfrak{g})$. Nikolayevsky defined such derivations and showed that they are unique up to automorphism. He called them *pre-Einstein derivations* because when the underlying Lie algebra is nilpotent, they are natural generalizations of the nilsoliton derivation used to define an Einstein solvable extension. Because they are purely algebraic objects of broader use, we prefer to call such a derivation a *Nikolayevsky derivation.* He also showed that if $\mathfrak{g}$ admits a soliton inner product, then the nilsoliton derivation is a scalar multiple of the Nikolayevsky derivation ([Nik11]). It follows from the proof of Theorem 1.1(a) of

[Nik11] that D^N is a Nikolayevsky derivation if and only if the condition in Equation (1) holds for all F in an $\mathbb{R}$-split torus $\mathfrak{t}^s$ containing D^N. Thus, it is elementary to find the Nikolayevsky derivations of a Lie algebra with real rank one:

Proposition 2.2. *[*[Nik11]*] Let $\mathfrak{g}$ be a nilpotent Lie algebra with real rank one. Let D be a nontrivial semisimple derivation with real eigenvalues. Then*

$$D^N = \frac{\mathrm{trace}(D)}{\mathrm{trace}(D^2)}\, D$$

is the unique Nikolayevsky derivation for $\mathfrak{g}$.

2.4. **Moduli spaces of soliton and nonsoliton nilpotent Lie algebras.** We need to describe the structure of the moduli space of nilpotent Lie algebras of dimension n. Let $\mathbb{R}^n$ be a real vector space of dimension n with basis $\mathcal{B} = \{x_i\}_{i=1}^n$. Suppose that $\mathbb{R}^n$ is endowed with a Lie bracket that defines a nilpotent Lie algebra structure on $\mathbb{R}^n$. The Lie bracket is equivalent to a skew-symmetric vector-valued bilinear map μ in the vector space $V = \wedge^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n$. The Jacobi identity and the nilpotency condition are polynomial constraints on the coefficients of μ in V with respect to the basis $\{x_i^* \wedge x_j^* \otimes x_k \,:\, 1 \leq i < j \leq n, 1 \leq k \leq n\}$, so we may identify each μ with an element $(\mu, \mathcal{B})$ of an affine subvariety X of V. We let $\mathfrak{n}_\mu$ denote the corresponding nilpotent Lie algebra. The general linear group $GL_n(\mathbb{R})$ acts on X by change of basis: for $\mu \in X$, the element $g \cdot \mu$ of X is defined by

$$(g \cdot \mu)(x, y) = g\mu(g^{-1}x, g^{-1}y),$$

for $x, y \in \mathbb{R}^n$. Two elements μ and ν of X define isomorphic Lie algebras $\mathfrak{n}_\mu$ and $\mathfrak{n}_\nu$ if and only if μ and ν are in the same $GL_n(\mathbb{R})$ orbit. The quotient $\mathcal{N}_n$ of X by this action is the moduli space of n-dimensional nilpotent Lie algebras. The equivalence class of μ is denoted by $\overline{\mu}$. We endow $\mathcal{N}_n$ with the quotient topology.

The properties of whether a Lie algebra $\mathfrak{n}_\mu$ for $\mu \in X$ is $\mathbb{N}$-graded, and whether it is indecomposable are both invariant under the $GL_n(\mathbb{R})$ action. Hence we may define the moduli space $\widetilde{\mathcal{N}}_n$ of $\mathbb{N}$-graded, indecomposable nilpotent Lie algebras to be the set of elements $\overline{\mu}$ of $\mathcal{N}_n$ so that $\mathfrak{n}_\mu$ is $\mathbb{N}$-graded and indecomposable. We use the subspace topology for $\widetilde{\mathcal{N}}_n$. The property of whether or not a Lie algebra $\mathfrak{n}_\mu$ admits a soliton inner product is also invariant under the $GL_n(\mathbb{R})$ action, so we may define the set $\mathrm{Nonsol}(n) \subseteq \widetilde{\mathcal{N}}_n$ to be the set of all $\overline{\mu}$ in $\widetilde{\mathcal{N}}_n$ such that $\mathfrak{n}_\mu$ does not admit a soliton inner product.

3. Continuous families of nonsoliton nilpotent Lie algebras in dimensions 8 and 9

3.1. A curve of nonsoliton nilpotent Lie algebras in dimension 8.

Definition 3.1. Let $s \in \mathbb{R}$, and let $\mathcal{B} = \{x_i\}_{i=1}^8$ be a fixed basis for $\mathbb{R}^8$. Define $\mathfrak{n}_s$ to be the nilpotent Lie algebra with underlying vector space $\mathbb{R}^8$ whose Lie algebra structure is determined by the bracket relations

$$\begin{array}{lll} [x_2, x_3] = e^{-s} x_4 & [x_1, x_3] = e^{s} x_5 & [x_1, x_2] = x_6 \\ [x_2, x_6] = e^{s} x_7 & [x_3, x_4] = e^{-s} x_7 & [x_1, x_6] = e^{-s} x_8 \\ [x_2, x_4] = x_8 & [x_3, x_5] = e^{s} x_8 & . \end{array} \tag{2}$$

The Jacobi Identity may be checked by confirming that there are no distinct choices of i, j and k so that $[x_i, [x_j, x_k]]$ is nonzero. (That there are no such choices may also be deduced from Theorem 7 of [Pay10]).

It is not hard to verify that for all s, the Lie algebra $\mathfrak{n}_s$ is three-step nilpotent of type $(3, 3, 2)$ and is indecomposable. For all s, we may write the vector space $\mathfrak{n}_s$ as the direct sum $\mathfrak{n}_s = V_1 \oplus V_2 \oplus V_3$, where V_1, V_2 and V_3 are the three steps

$$\begin{aligned} V_1 &= \operatorname{span}\{x_1, x_2, x_3\} \\ V_2 &= \operatorname{span}\{x_4, x_5, x_6\} \\ V_3 &= \operatorname{span}\{x_7, x_8\}. \end{aligned}$$

Define the derivation $D : \mathfrak{n}_s \to \mathfrak{n}_s$ by

$$D(x) = kx, \quad \text{if } x \in V_k, \text{ for } k = 1, 2, 3.$$

The eigenspaces V_1, V_2, V_3 for D define an $\mathbb{N}$-grading of $\mathfrak{n}_s$.

Now we describe the derivation algebra of a typical nilpotent Lie algebra in the family defined in Definition 3.1.

Proposition 3.2. *Let $\mathfrak{n}_s$ be the nilpotent Lie algebra as defined in Definition 3.1, for any fixed s in $\mathbb{R}$. Then the derivation algebra $\operatorname{Der}(\mathfrak{n}_s)$ of $\mathfrak{n}_s$ is a 16-dimensional solvable algebra with real rank one. The derivation algebra decomposes as $\operatorname{Der}(\mathfrak{n}_s) = \mathbb{R}\, D + \mathfrak{m}$, where D is the derivation D defined above and $\mathfrak{m}$ is the nilradical. The derivation $D^N = \frac{5}{11} D$ is a Nikolayevsky derivation of $\mathfrak{n}_s$.*

Proof. The derivation algebra of $\mathfrak{n}_s$ is a subspace of $\operatorname{End}(\mathfrak{n}_s)$. The subspace may be described by a system of 8^3 linear equations in 8^2 unknowns, where the coefficients of the linear equations depend on the structure constants for $\mathfrak{n}_s$. (See Section 1.9 of [dG00].) The structure constants for $\mathfrak{n}_s$ depend on the parameter s. Using Matlab to solve this

system symbolically, we found that for any s, the solution space to the linear system is 16-dimensional and is spanned by the derivation D and 15 nilpotent derivations that span the nilpotent subalgebra $\mathfrak{m}$. Hence any semisimple derivation of $\mathfrak{n}_s$ is a scalar multiple of D and the real rank of $\mathfrak{n}_s$ is one. By Proposition 2.2, $D^N = \frac{5}{11}D$ is a Nikolayevsky derivation of $\mathfrak{n}_s$. $\square$

Now we show that none of the nilpotent Lie algebras in the family defined in Definition 3.1 are soliton.

Theorem 3.3. *Suppose that $\mathfrak{n}_s$ is a nilpotent Lie algebra as defined in Definition 3.1. Then $\mathfrak{n}_s$ does not admit a soliton inner product.*

Proof. Fix s in $\mathbb{R}$ and suppose that $\mathfrak{n}_s$ admits a soliton inner product Q. With respect to the basis $\mathcal{B}$, the Gram matrix U is

$$U = \begin{bmatrix} 3 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 3 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 3 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 3 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 3 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 3 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 3 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 3 \end{bmatrix}. \tag{3}$$

The solution space to the linear equation $U\boldsymbol{v} = [1]$ is $\{\boldsymbol{v}_0 + t\boldsymbol{v}_1 \,:\, t \in \mathbb{R}\}$, where

$$\boldsymbol{v}_0 = \tfrac{1}{11}(1,1,3,2,2,2,0,2)^T, \quad \text{and}$$
$$\boldsymbol{v}_1 = (-1,1,0,1,-1,-1,0,1)^T.$$

For all such solutions $\boldsymbol{v} = (v_i)$, $v_7 = 0$. By Theorem 2.1, $\mathfrak{n}_s$ does not admit a soliton inner product. $\square$

We will show in Theorem 4.5 that no two nilpotent Lie algebras in the family defined in Definition 3.1 are isomorphic.

3.2. **A curve of nonsoliton nilpotent Lie algebras in dimension 9.** Now we define a one-parameter family of 9-dimensional nilpotent Lie algebras, similar to the family of 8-dimensional nilpotent Lie algebras defined in the previous section.

Definition 3.4. Let $\mathcal{B} = \{x_i\}_{i=1}^9$ be a basis for $\mathbb{R}^9$. Let s be a real number. Let $\mathfrak{n}_s$ be the Lie algebra with underlying vector space $\mathbb{R}^9$

whose Lie algebra structure is determined by the Lie bracket relations

$$\begin{array}{lll} [x_2, x_3] = e^{4s} x_4 & [x_1, x_3] = e^{-3s} x_5 & [x_1, x_2] = e^{-s} x_6 \\ [x_2, x_6] = e^{-4s} x_7 & [x_3, x_4] = e^{4s} x_7 & [x_1, x_6] = e^{4s} x_8 \\ [x_2, x_4] = x_8 & [x_3, x_5] = e^{-4s} x_8 & [x_3, x_6] = e^{-s} x_9 \\ & [x_2, x_5] = e^{s} x_9. & \end{array}$$

The Jacobi Identity may be confirmed by direct computation, noting that the only time $[[x_i, x_j], x_k]$ is nontrivial for distinct i, j, k is when $\{i, j, k\} = \{1, 2, 3\}$. (The latter fact follows from Theorem 7 of [Pay10].)

Each Lie algebra $\mathfrak{n}_s$ is three-step nilpotent of type $(3, 3, 3)$. We may write the vector space $\mathbb{R}^9 \cong \mathfrak{n}_s$ as the direct sum $\mathbb{R}^9 = V_1 \oplus V_2 \oplus V_3$, where V_1, V_2 and V_3 are the three steps for any $\mathfrak{n}_s$:

$$\begin{aligned} V_1 &= \operatorname{span}\{x_1, x_2, x_3\} \\ V_2 &= \operatorname{span}\{x_4, x_5, x_6\} \\ V_3 &= \operatorname{span}\{x_7, x_8, x_9\}. \end{aligned}$$

A derivation D of $\mathfrak{n}_s$ is defined by

$$D(x) = kx, \qquad \text{if } x \in V_k, \text{ for } k = 1, 2, 3.$$

The eigenspaces V_1, V_2, V_3 for D define an $\mathbb{N}$-grading of $\mathfrak{n}_s$.

Proposition 3.5. *Let $s \in \mathbb{R}$, and let $\mathfrak{n}_s$ be as defined in Definition 3.4. The derivation algebra $\operatorname{Der}(\mathfrak{n}_s) = \mathbb{R}\, D + \mathfrak{m}$ of $\mathfrak{n}_s$, is 19-dimensional and solvable, with 18-dimensional nilradical $\mathfrak{m}$. The real rank of $\mathfrak{n}_s$ is one, and $D^N = \frac{9}{14} D$ is a Nikolayevsky derivation of $\mathfrak{n}_s$.*

The proof of the proposition is analogous to that of Proposition 3.2, so we do not include it. Now we show that none of the Lie algebras defined in Definition 3.4 are soliton.

Theorem 3.6. *Let $s \in \mathbb{R}$, and let $\mathfrak{n}_s$ be a nilpotent Lie algebra as defined in Definition 3.4. Then $\mathfrak{n}_s$ does not admit a soliton inner product.*

Proof. The proof is the same as that of Theorem 3.3, except that

$$U = \begin{bmatrix} 3 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 3 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & -1 \\ 1 & 1 & 3 & 0 & 0 & 0 & 1 & 0 & -1 & 1 \\ 1 & 0 & 0 & 3 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 3 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 3 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 3 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 3 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 & 1 & 0 & 1 & 3 & 1 \\ 1 & -1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 3 \end{bmatrix} \tag{4}$$

and the solution space to $U\boldsymbol{v} = [1]$ is $\{\boldsymbol{v}_0 + s\boldsymbol{v}_1 + t\boldsymbol{v}_2 \,:\, s, t \in \mathbb{R}\}$, where

$$\boldsymbol{v}_0 = \tfrac{1}{161}(5, 25, 39, 18, 28, 28, 0, 18, 16, 30)^T.$$
$$\boldsymbol{v}_1 = (-1, 1, 0, 1, -1, -1, 0, 1, 0, 0)^T$$
$$\boldsymbol{v}_2 = (0, 1, -1, 0, 0, 0, 0, 0, -1, 1)^T$$

All vectors $\boldsymbol{v} = (v_i)$ in the solution space have $v_7 = 0$. □

We will show in Theorem 4.5 of Section 4 that if $\mathfrak{n}_s$ and $\mathfrak{n}_t$ are nilpotent Lie algebras in the family defined in Definition 3.4, $\mathfrak{n}_s$ and $\mathfrak{n}_t$ are isomorphic if and only if $s = t$.

4. Constructions of curves of nonsoliton nilpotent Lie algebras in higher dimensions

4.1. **Examples in dimensions $n \geq 10$.** We describe how we construct the higher-dimensional examples from the 8- and 9- dimensional ones already defined.

Definition 4.1. In each even dimension $8 + 2k$, where $k \in \mathbb{N}$, a family of Lie algebras $\mathfrak{n}_s^{8+2k}$ is defined as follows. For each $s \in \mathbb{R}$, let $\mathfrak{n}_s$ be the 8-dimensional nilpotent Lie algebra defined in Definition 3.1. For $k \geq 1$, the $(8 + 2k)$-dimensional nilpotent Lie algebra $\mathfrak{n}_s^{8+2k}$ is represented with respect to a basis $\{x_i\}_{i=1}^8 \cup \{y_i\}_{i=1}^{2k}$ so that the bracket relations defining the Lie algebra structure are those listed for $\mathfrak{n}_s$ in Equation (2) of Definition 3.1, along with the k additional generating bracket relations

$$[y_i, y_{2k-i}] = x_8, \quad \text{for } i = 1, \dots, k.$$

If necessary, we may let $x_{m+i} = y_i$ for $i = 1, \dots, 2k$ to define an ordering the basis $\mathcal{B} = \{x_i\}_{i=1}^m \cup \{y_i\}_{2k} = \{x_i\}_{i=1}^{m+2k}$ in accord with the subscripts on the x_i's. When $k = 0$, we let $\mathfrak{n}_s^{8+2k} = \mathfrak{n}_s$.

In odd dimensions $9+2k$, the Lie algebras $\mathfrak{n}_s^{9+2k}$ are defined similarly. For any $s \in \mathbb{R}$, let $\mathfrak{n}_s$ be the 9-dimensional nilpotent Lie algebra defined in Definition 3.1. For $k \geq 1$, the $(9+2k)$-dimensional nilpotent Lie algebra $\mathfrak{n}_s^{9+2k}$ is represented with respect to the basis $\{x_i\}_{i=1}^9 \cup \{y_i\}_{i=1}^{2k}$ with the bracket relations for $\mathfrak{n}_s$ in Definition 3.4, along with the additional bracket relations

$$[y_i, y_{2k-i}] = x_9, \quad \text{for } i = 1, \ldots, k.$$

When $k = 0$, we let $\mathfrak{n}_s^{9+2k} = \mathfrak{n}_s$.

One may confirm without too much effort that the Jacobi Identity holds for all of these product structures. For any Lie algebra $\mathfrak{n}_s^{8+2k}$ or $\mathfrak{n}_s^{9+2k}$, with $k \in \mathbb{N}$, the only time a double bracket $[[x_i, x_j], x_k]$ vanishes for distinct i, j, k is when $\{i, j, k\} = \{1, 2, 3\}$. (This follows from Theorem 7 of [Pay10].) One may also verify that for all $s \in \mathbb{R}$ and $k \geq 0$, the Lie algebra $\mathfrak{n}_s^{8+2k}$ is three-step nilpotent of type $(2k+3, 3, 2)$, and the Lie algebra $\mathfrak{n}_s^{9+2k}$ is three-step nilpotent of type $(2k+3, 3, 3)$.

Now fix $\mathfrak{n}_s^{m+2k}$, where $m = 8$ or 9, and $k \in \mathbb{N}$. Define the subspaces V_2, V_3, V_4 and V_6 by

$$\begin{aligned} V_2 &= \operatorname{span}\{x_1, x_2, x_3\} \\ V_3 &= \operatorname{span}\{y_1, \ldots, y_{2k}\} \\ V_4 &= \operatorname{span}\{x_4, x_5, x_6\} \\ V_6 &= \operatorname{span}\{x_7, \ldots, x_m\}. \end{aligned} \tag{5}$$

When $k = 0$, we let V_2, V_4, and V_6 be as defined above, and we let $V_3 = \{0\}$. Then $\mathfrak{n}_s^{m+2k} = V_2 \oplus V_3 \oplus V_4 \oplus V_6$ for all $k \geq 0$.

For $k \geq 0$, define the derivation $D : \mathfrak{n}_s^{m+2k} \to \mathfrak{n}_s^{m+2k}$ by

$$D(x) = kx, \quad \text{if } x \in V_k, \text{ for } k = 2, 3, 4, 6. \tag{6}$$

Because D is a derivation, for $m = 8$ or 9, and $k \geq 0$, the eigenspaces V_i define an $\mathbb{N}$-grading of the Lie algebra $\mathfrak{n}_s^{m+2k} : [V_i, V_j] \subseteq V_{i+j}$, where $V_k = 0$ when $k \notin \{2, 3, 4, 6\}$.

We will need the following lemma later.

Lemma 4.2. *Let $\mathfrak{g}$ be a Lie algebra, and let $\mathfrak{i}$ and $\mathfrak{j}$ be ideals in $\mathfrak{g}$ such that $\mathfrak{g}$ is the sum (not necessarily a direct sum) of $\mathfrak{i}$ and $\mathfrak{j}$, and $[\mathfrak{i}, \mathfrak{j}] = 0$.*

Let $\pi : \mathfrak{g} \to \mathfrak{g}$ denote a projection map from $\mathfrak{g}$ to $\mathfrak{i}$; i.e., an endomorphism such that $\pi|_{\mathfrak{i}} = Id|_{\mathfrak{i}}$ and $\pi(\mathfrak{g}) = \mathfrak{i}$. Let D be a derivation of $\mathfrak{g}$. Then the restriction of $\pi_1 \circ D$ to $\mathfrak{i}$ is a derivation of $\mathfrak{i}$.

Conversely, if $D_1 : \mathfrak{i} \to \mathfrak{i}$ is a derivation of $\mathfrak{i}$, $D_2 : \mathfrak{j} \to \mathfrak{j}$ is a derivation of $\mathfrak{j}$, and $D_1(z) = D_2(z)$ for all $z \in \mathfrak{i} \cap \mathfrak{j}$, then

$$D(z) = \begin{cases} D_1(z) & z \in \mathfrak{i} \\ D_2(z) & z \in \mathfrak{j} \end{cases}$$

is a derivation of $\mathfrak{g}$.

The solvable radical of $\mathrm{Der}(\mathfrak{g})$ *contains the solvable radical of* $\mathrm{Der}(\mathfrak{i})$.

Proof. There exists a basis $\{x_i\}_{i=1}^m \cup \{y_j\}_{j=1}^d$ for $\mathfrak{g}$ such that

$$\mathfrak{i} = \mathrm{span}\{x_i\}_{i=1}^m \qquad \text{and} \qquad \mathfrak{j} < \mathrm{span}\{y_j\}_{j=1}^d + \mathfrak{z},$$

and the projection $\pi : \mathfrak{g} \to \mathfrak{i}$ from $\mathfrak{g}$ to $\mathfrak{i}$ is given by

$$\pi(x_i) = x_i, \quad \text{for } i = 1, \ldots, m, \text{ and}$$
$$\pi(y_j) = 0 \quad \text{for } j = 1, \ldots, d.$$

Because D is a derivation, for all $i, j = 1, \ldots, m$,

$$D([x_i, x_j]) = [D(x_i), x_j] + [x_i, D(x_j)]. \tag{7}$$

As $[y_l, x_j] = 0$ for all $l = 1, \ldots, d$ and $j = 1, \ldots, m$, we get

$$D([x_i, x_j]) = [(\pi_1 \circ D)(x_i), x_j] + [x_i, (\pi_1 \circ D)(x_j)] \tag{8}$$

for all $i, j = 1, \ldots, m$. The vectors on the right side of Equation (7) are in $\mathfrak{i}$, hence the vector on the left side is also in $\mathfrak{i}$, so we have

$$(\pi \circ D)([x_i, x_j]) = [(\pi \circ D)(x_i), x_j] + [x_i, (\pi \circ D)(x_j)]$$

for all vectors x_i and x_j in the basis $\{x_i\}_{i=1}^m$ of $\mathfrak{i}$. Hence, the restriction of $\pi \circ D$ to $\mathfrak{i}$ is a derivation of $\mathfrak{i}$.

To prove the converse, note that Equation (8) holds for all $x_i \in \mathfrak{i}$ and $x_j \in \mathfrak{j}$ due to the fact that $[\mathfrak{i}, \mathfrak{j}] = 0$, while it holds if either both x_i and x_j are in $\mathfrak{i}$ or both x_i and x_j are in $\mathfrak{j}$ because D_1 and D_2 are derivations of $\mathfrak{i}$ and $\mathfrak{j}$ respectively. □

The next proposition describes the Nikolayevsky derivations of the Lie algebras defined in Definition 4.1.

Proposition 4.3. *Let $m = 8$ or 9, and let $k \geq 0$. For any $s \in \mathbb{R}$, let $\mathfrak{n}_s^{m+2k}$ an $(m+2k)$-dimensional nilpotent Lie algebra as defined in Definition 4.1. The derivation $D^N = \frac{m+k-3}{6m+3k-26} D$, where D is defined in Equation (6), is a Nikolayevsky derivation of $\mathfrak{n}_s^{m+2k}$.*

Proof. Fix s, and let $\mathfrak{n}_s^{m+2k}$ be an $(m+2k)$-dimensional nilpotent Lie algebra as defined in Definition 4.1. Then the subspace $\mathfrak{m} = \mathrm{span}\{x_i\}_{i=1}^m$ is an ideal. If $m = 8$, the ideal $\mathfrak{m}$ is isomorphic to the Lie algebra $\mathfrak{n}_s$ defined in Definition 3.1, and if $m = 9$, then $\mathfrak{m}$ is isomorphic to the

Lie algebra $\mathfrak{n}_s$ defined in Definition 3.4. Let $\mathfrak{h} = \mathrm{span}(\{x_m\} \cup \{y_j\}_{j=1}^{2k})$. The subspace $\mathfrak{h}$ is an ideal isomorphic to the Heisenberg algebra of dimension $2k+1$.

Define the derivation D_0 of $\mathfrak{m}$ by

$$D_0(w) = \begin{cases} 2\lambda w & w \in V_2 = \mathrm{span}\{x_i\}_{i=1}^{3} \\ 3\lambda w & w \in V_3 = \mathrm{span}\{y_j\}_{j=1}^{2k} \\ 4\lambda w & w \in V_4 = \mathrm{span}\{x_i\}_{i=4}^{6} \\ 6\lambda w & w \in V_6 = \mathrm{span}\{x_i\}_{i=7}^{m} \end{cases}, \tag{9}$$

where $\lambda = \frac{m+k-3}{6m+3k-26}$.

We want to show that D_0 is a Nikolayevsky derivation by showing that

$$\mathrm{trace}(D_0 \circ F) = \mathrm{trace}(F)$$

for all F in $\mathrm{Der}(\mathfrak{n}_s^{m+2k})$.

Let $\pi_{\mathfrak{m}} : \mathfrak{n}_s^{m+2k} \to \mathfrak{m}$ and $\pi_{\mathfrak{h}} : \mathfrak{n}_s^{m+2k} \to \mathfrak{h}$ be the projection maps defined by the basis $\mathrm{span}\{x_i\}_{i=1}^{m} \cup \{y_j\}_{j=1}^{2k}$. By Lemma 4.2, the restriction $\pi_{\mathfrak{m}} \circ F|_{\mathfrak{m}}$ of F to $\mathfrak{m}$ is a derivation of $\mathfrak{m}$. By Proposition 3.2 (if $m = 8$) or Proposition 3.5 (if $m = 9$), the restriction of F to $\mathfrak{m}$ is a nonzero scalar multiple of the restriction of D_0 to $\mathfrak{m}$. Similarly, the restriction $\pi_{\mathfrak{h}} \circ F|_{\mathfrak{h}}$ of F to $\mathfrak{h}$ is a derivation F_1 of $\mathfrak{h}$. Thus, we may write F as the sum of derivations $F = \hat{F} + (F - \hat{F})$, where

$$\begin{aligned} \hat{F}(w) &= \begin{cases} \pi_{\mathfrak{m}} \circ F|_{\mathfrak{m}}(w) & w \in \mathfrak{m} = \mathrm{span}\{x_1, \ldots, x_m\} \\ \pi_{\mathfrak{h}} \circ F|_{\mathfrak{h}}(w) & w \in \mathrm{span}\{y_j\}_{j=1}^{2k}, \end{cases} \\ &= \begin{cases} cD_0(w) & w \in \mathfrak{m} = \mathrm{span}\{x_1, \ldots, x_m\} \\ F_1(w) & w \in \mathrm{span}\{y_j\}_{j=1}^{2k}, \end{cases} \end{aligned}$$

for some $c \in \mathbb{R}$, and $F_1 : \mathfrak{h} \to \mathfrak{h}$ is a derivation of the ideal $\mathfrak{h}$.

The derivation $\hat{F}$ fixes the ideals $\mathfrak{m}$ and $\mathfrak{h}$, and is block diagonal when represented with respect to the basis. The restriction of $\hat{F}$ to $\mathfrak{h}$ is the derivation F_1 of $\mathfrak{h}$. The derivation $F - \hat{F}$ maps $\mathfrak{m} = \mathrm{span}\{x_1, \ldots, x_m\}$ into $\mathrm{span}\{y_j\}_{j=1}^{2k}$ and it maps $\mathrm{span}\{y_j\}_{j=1}^{2k}$ into $\mathfrak{m} = \mathrm{span}\{x_1, \ldots, x_m\}$, and when represented with respect to the basis $\mathcal{B}$ in block form has 0 blocks along the diagonal.

Using the definition of D_0 we see that $\mathrm{trace}(D_0 \circ (F - \hat{F})) = 0$; hence,

$$\mathrm{trace}(D_0 \circ \hat{F}) = \mathrm{trace}(D_0 \circ F).$$

In addition, $\mathrm{trace}\, \hat{F} = \mathrm{trace}\, F$. Thus, in order to show that $\mathrm{trace}(D_0 \circ F) = \mathrm{trace}(F)$ for all derivations F, it suffices to show that $\mathrm{trace}(D_0 \circ \hat{F}) = \mathrm{trace}(\hat{F})$ for all derivations F.

We will compute $\operatorname{trace}(D_0 \circ \hat{F})$ and $\operatorname{trace}(\hat{F})$ directly. But first we will prove the following claim: $k \cdot \operatorname{trace} \hat{F}|_{\mathbb{R}x_m} = \operatorname{trace} \hat{F}|_{V_3}$. It is not hard to confirm the basic fact that the derivation $D^N_{\mathfrak{h}} : \mathfrak{h} \to \mathfrak{h}$ defined by

$$D^N_{\mathfrak{h}}(v) = \begin{cases} \frac{k+1}{k+2}\, v & v \in V_3 = \operatorname{span}\{y_l\}_{l=1}^{2k} \\ 2\, \frac{k+1}{k+2}\, v & v \in \mathbb{R}x_m \end{cases}$$

is a Nikolayevsky derivation for the ideal $\mathfrak{h} \cong \mathfrak{h}_{2k+1}$. By the defining property of the Nikolayevsky derivation $D^N_{\mathfrak{h}}$ for $\mathfrak{h}$,

$$\operatorname{trace}(D^N_{\mathfrak{h}} \circ \hat{F}|_{\mathfrak{h}}) = \operatorname{trace}(\hat{F}|_{\mathfrak{h}}),$$

which becomes

$$\left(\frac{k+1}{k+2}\right)\Big(\operatorname{trace}(\hat{F}|_{V_3}) + 2\operatorname{trace}(\hat{F}|_{\mathbb{R}x_m})\Big) = \operatorname{trace}(\hat{F}|_{V_3}) + \operatorname{trace}(\hat{F}|_{\mathbb{R}x_m}).$$

After simple arithmetic manipulations we get the desired equality $k \cdot \operatorname{trace} \hat{F}|_{\mathbb{R}x_8} = \operatorname{trace} \hat{F}|_{V_3}$. Then

$$\operatorname{trace}(\hat{F}|_{V_3}) = k \cdot \operatorname{trace} \hat{F}|_{\mathbb{R}x_8} = k \cdot \operatorname{trace}(cD_0)|_{\mathbb{R}x_8} = 6c\lambda k. \tag{10}$$

We return to the computation of $\operatorname{trace}(D_0 \circ \hat{F})$ and $\operatorname{trace}(\hat{F})$, using the definitions of D_0 and $\hat{F}$, finding

$$\begin{aligned} \operatorname{trace}(D_0 \circ \hat{F}) &= \operatorname{trace}\Big((D_0 \circ \hat{F})|_{\mathfrak{m}}\Big) + \operatorname{trace}\Big((D_0 \circ \hat{F})|_{V_3}\Big) \\ &= c \operatorname{trace}\left(D_0^2|_{\mathfrak{m}}\right) + \operatorname{trace}\Big((3\lambda \operatorname{Id}_{V_3} \circ \hat{F})|_{V_3}\Big), \end{aligned}$$

where Id_{V_3} denotes the identity map on V_3. Continuing, using the definition of D_0 we get

$$\begin{aligned} \operatorname{trace}(D_0 \circ \hat{F}) &= c(3 \cdot 4\lambda^2 + 3 \cdot 16\lambda^2 + (m-6) \cdot 36\lambda^2) + 3\lambda \operatorname{trace}(\hat{F}|_{V_3}) \\ &= c\lambda^2(36m - 156) + 3\lambda \cdot 6c\lambda k \quad \text{from Equation (10)} \\ &= c\lambda(36m - 156 + 18k) \cdot \lambda, \\ &= 6\lambda c(6m + 3k - 26)\frac{m+k-3}{6m+3k-26} \quad \text{by definition of } \lambda \\ &= 6\lambda c\,(m + k - 3), \end{aligned}$$

while parallelly,

$$\begin{aligned} \operatorname{trace}(\hat{F}) &= \operatorname{trace}(\hat{F}|_{\mathfrak{m}}) + \operatorname{trace}(\hat{F}|_{V_3}) \\ &= 3 \cdot 2c\lambda + 3 \cdot 4c\lambda + (m-6) \cdot 6c\lambda + \operatorname{trace}(\hat{F}|_{V_3}) \\ &= (6c\lambda m - 18c\lambda) + 6c\lambda k \\ &= 6\lambda c\,(m + k - 3). \end{aligned}$$

Thus, D_0 is a Nikolayevsky derivation for $\mathfrak{n}_s^{m+2k}$ as claimed. $\square$

Now we prove a technical lemma about isomorphisms of the algebras which we have defined.

Lemma 4.4. *Let $m = 8$ or 9, let $s \in \mathbb{R}$, and let $k \geq 0$. Let $\mathfrak{n}_s^{m+2k}$ be a $(m+2k)$-dimensional nilpotent Lie algebra as defined in Definition 4.1, and let V_3, V_4 and V_6 be as defined in Equation (5). Suppose that $\phi : \mathfrak{n}_s^{m+2k} \to \mathfrak{n}_s^{m+2k}$ is an isomorphism. Then*

(1) *If $m = 8$, then ϕ maps the subspaces $\mathbb{R}x_1 \oplus V_3 \oplus V_4 \oplus V_6$ and $\mathbb{R}x_5 \oplus V_3 \oplus V_6$ to themselves.*
(2) *If $m = 9$, then ϕ maps the subspaces $\mathbb{R}x_1 \oplus V_4 \oplus V_6$ and $\mathbb{R}x_4 \oplus \mathbb{R}x_5 \oplus V_4 \oplus V_6$ to themselves.*

Proof. The two subspaces $\mathbb{R}\,x_1 \oplus V_3 \oplus V_4 \oplus V_6$ and $\mathbb{R}x_4 \oplus \mathbb{R}x_5 \oplus V_3 \oplus V_6$ are fixed by isomorphisms because each may be uniquely characterized by algebraic properties that are preserved under isomorphisms.

We claim that when $m = 8$, the subset

$$S = \{b_1 x_1 + v \,:\, b_1 \neq 0, v \in V_3 \oplus V_4 \oplus V_6\}$$

is the set of all elements x such that ad_x has rank 3, where ad_x is the adjoint map for either $\mathfrak{n}_s^{m+2k}$ or $\mathfrak{n}_t^{m+2k}$.

For example, when $k = 1$, if $x = \sum_{i=1}^{8} b_i x_i + c_1 y_1 + c_2 y_2$, then with respect to the usual basis $\mathcal{B}$, the adjoint map ad_x for $\mathfrak{n}_s^{10}$ is represented by the matrix

$$[\mathrm{ad}_x]_{\mathcal{B}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -e^{-s}b_3 & e^{-s}b_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -e^{s}b_3 & 0 & e^{s}b_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -b_2 & b_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -e^{s}b_6 & -e^{-s}b_4 & e^{-s}b_3 & 0 & e^{s}b_2 & 0 & 0 & 0 & 0 \\ -e^{-s}b_6 & -b_4 & -e^{s}b_5 & b_2 & e^{s}b_3 & e^{-s}b_1 & 0 & 0 & -c_2 & c_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The rank of the submatrix

$$M_1 = \begin{bmatrix} 0 & -e^{-s}b_3 & e^{-s}b_2 \\ -e^{s}b_3 & 0 & e^{s}b_1 \\ -b_2 & b_1 & 0 \end{bmatrix}$$

is two if and only if $(b_1, b_2, b_3) \neq (0, 0, 0)$, and is zero otherwise.

Therefore if the rank of ad_x is 3, then $(b_1, b_2, b_3) \neq (0, 0, 0)$. But if $b_2 \neq 0$ or $b_3 \neq 0$, then the minor

$$\begin{bmatrix} e^{-s}b_3 & 0 & e^{s}b_2 \\ b_2 & e^{s}b_3 & e^{-s}b_1 \end{bmatrix}$$

has rank two. The block form of the matrix then forces the rank of the larger matrix to be at least four, a contradiction. Hence $b_2 = b_3 = 0$, and $x \in S$. Conversely, if $b_1 \neq 0$ and $b_2 = b_3 = 0$, then rows 5, 6 and 8 form a basis for the row space of the matrix.

Since isomorphisms preserve the rank of ad_x, S is invariant under isomorphisms. By continuity, an isomorphism fixes the closure of S, $\mathbb{R}_1 x_1 \oplus V_3 \oplus V_4 \oplus V_6$.

The subspace $\mathbb{R}x_5 \oplus V_4 \oplus V_6$ can be characterized algebraically as the closure of the set of nonzero elements x in the commutator ideal $V_4 \oplus V_6$ such that the rank of ad_x is 1. This is seen by letting b_1, b_2, b_3, c_1 and c_2 equal zero in the matrix representing ad_x .

Thus we have shown that $\mathbb{R}x_1 \oplus V_3 \oplus V_4 \oplus V_6$ and $\mathbb{R}x_5 \oplus V_3 \oplus V_6$ are preserved by isomorphisms, establishing Part (1) of the lemma in the case that $m = 10$. The same arguments apply in higher even dimensions $8 + 2k > 10$.

Now suppose the $m = 9$. Let

$$S = \{b_1 x_1 + v \,:\, b_1 \neq 0, v \in V_4 \oplus V_6\}.$$

We assert that S is the set of all elements x such that ad_x has rank 3, and x is not in the centralizer of the commutator ideal. The commutator ideal is $V_4 \oplus V_6$ and its centralizer is

$$\mathfrak{z}(V_4 \oplus V_6) = V_3 \oplus V_4 \oplus V_6.$$

If $x = \sum_{i=1}^{9} b_i x_i + c_1 y_1 + c_2 y_2$, then adjoint map ad_x for $\mathfrak{n}_s^{11}$ is represented by the matrix

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -e^{4s}b_3 & e^{4s}b_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -e^{-3s}b_3 & 0 & e^{-3s}b_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -e^{-a}b_2 & e^{-a}b_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -e^{4s}b_6 & -e^{4s}b_4 & e^{4s}b_3 & 0 & e^{-4s}b_2 & 0 & 0 & 0 & 0 & 0 \\ -e^{4s}b_6 & -b_4 & -e^{-4s}b_5 & b_2 & e^{-4s}b_3 & e^{4s}b_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -e^{s}b_5 & -e^{-s}b_6 & 0 & e^{s}b_2 & e^{-s}b_3 & 0 & 0 & 0 & -c_2 & c_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The rank of the submatrix

$$\begin{bmatrix} 0 & -e^{4s}b_3 & e^{4s}b_2 \\ -e^{-3s}b_3 & 0 & e^{-3s}b_1 \\ -e^{-a}b_2 & e^{-a}b_1 & 0 \end{bmatrix}$$

is two if and only if $(b_1, b_2, b_3) \neq 0$ and is zero otherwise.

Assume that x is not in $\mathfrak{z}(V_4 \oplus V_6)$, and that the rank of ad_x is three. Since x is not in the centralizer of the commutator, $(b_1, b_2, b_3) \neq (0, 0, 0)$. If $b_2 \neq 0$ or $b_3 \neq 0$, then the minor

$$\begin{bmatrix} e^{4s}b_3 & 0 & e^{-4s}b_2 \\ b_2 & e^{-4s}b_3 & e^{4s}b_1 \\ 0 & e^{s}b_2 & e^{-s}b_3 \end{bmatrix}.$$

has rank 2 or more, forcing the larger matrix to have rank greater than three, a contradiction. Therefore $b_2 = b_3 = 0$. Substituting these values into the larger matrix, we see that rows 5, 6 and 8 are independent, and row 9 will not being in their span unless $c_1 = c_2 = 0$. Hence, $x \in S$.

Conversely, if $x \in S$, then $b_1 \neq 0$, $b_2 = b_3 = 0$, and $c_1 = c_2 = 0$. Since $b_1 \neq 0$, the vector x is not in the centralizer of the commutator ideal. After substituting the zero values into the large matrix, we see that rows 7 and 9 are in the span of the independent rows 5, 6 and 8. Hence the rank of ad_x is 3.

Thus we have shown that S can be characterized as the set of all x such that ad_x is not in the centralizer of the commutator ideal and ad_x has rank 3. Therefore $\phi(S) = S$, and ϕ preserves the closure $\mathbb{R}x_1 \oplus V_4 \oplus V_6$.

We have just shown that an isometry ϕ preserves $W = \mathbb{R}x_1 \oplus V_4 \oplus V_6$. Therefore ϕ will also map the centralizer of W,

$$\mathfrak{z}(W) = \{x \in \mathfrak{n}_s^{m+2k} \,:\, [x, y] = 0 \text{ for all } y \in \mathbb{R}x_1 \oplus V_4 \oplus V_6\}$$

of W for $\mathfrak{n}_s^{m+2k}$ to the centralizer of $\phi(W) = W$ for $\mathfrak{n}_t^{m+2k}$. But

$$\mathfrak{z}(W) = \mathbb{R}x_4 \oplus \mathbb{R}x_5 \oplus V_3 \oplus V_6.$$

Therefore, ϕ preserves $\mathbb{R}x_4 \oplus \mathbb{R}x_5 \oplus V_3 \oplus V_6$ as claimed.

This we have shown that Part (2) holds in dimension 11. The same arguments apply in odd dimensions $9 + 2k$ greater than 11. □

Now we are ready to show that for fixed m and k, no distinct two Lie algebras in the family $\mathfrak{n}_s^{m+2k}$, $s \in \mathbb{R}$ are isomorphic.

Theorem 4.5. *Let $m = 8$ or 9, let $s, t \in \mathbb{R}$, and let $k \geq 0$. Let $\mathfrak{n}_s^{m+2k}$ and $\mathfrak{n}_t^{m+2k}$ be two $(m+2k)$-dimensional nilpotent Lie algebras as defined in Definition 4.1. Then $\mathfrak{n}_s^{m+2k}$ and $\mathfrak{n}_t^{m+2k}$ are isomorphic if and only if $s = t$.*

Proof. Suppose that $\phi : \mathfrak{n}_s^{m+2k} \to \mathfrak{n}_t^{m+2k}$ is an isomorphism. We view $\mathfrak{n}_s^{m+2k}$ and $\mathfrak{n}_t^{m+2k}$ as the same vector space endowed with different Lie brackets. Let V_i denote the eigenspace for the derivation D with eigenvalue i as in Equation (5). Recall that eigenspaces V_i, where $i = 2, 3, 4, 6$, define an $\mathbb{N}$-grading of $\mathfrak{n}_s^{m+2k} : [V_i, V_j] \subseteq V_{i+j}$, where $V_l = 0$ when $l \in \mathbb{N} \setminus \{2, 3, 4, 6\}$, and $V_3 = \{0\}$ when $k = 0$. In particular, we know that

$$[V_3 \oplus V_4 \oplus V_6, \mathfrak{n}_t^{m+2k}] \subseteq V_6,$$
$$[V_3 \oplus V_4 \oplus V_6, V_4 \oplus V_6] = \{0\},$$

and that V_6 is central.

By Lemma 4.4, Part (1), the isomorphism ϕ maps the subspace $\mathbb{R}x_1$ into the subspace $\mathbb{R}x_1 \oplus V_3 \oplus V_4 \oplus V_6$. Therefore we may write

$$\phi(x_1) = a_{11}x_1 + v_1$$

for some vector $v_1 \in V_3 \oplus V_4 \oplus V_6$ and some $a_{11} \in \mathbb{R}$. As ϕ is an isomorphism, $a_{11} \neq 0$; were a_{11} to vanish, $\phi(x_1)$ would be in the centralizer of the commutator while x_1 was not. We write

$$\begin{aligned} \phi(x_2) &= a_{12}x_1 + a_{22}x_2 + a_{32}x_3 + v_2 \\ \phi(x_3) &= a_{13}x_1 + a_{23}x_2 + a_{33}x_3 + v_3 \end{aligned} \tag{11}$$

for scalars $a_{12}, a_{22}, a_{32}, a_{13}, a_{23}$ and a_{33}, and vectors v_2 and v_3 in $V_3 \oplus V_4 \oplus V_6$.

To complete the proof, we will consider the cases $m = 8$ and $m = 9$ separately. First suppose that $m = 8$. The first defining relation $[x_2, x_3] = e^{-s}x_4$ yields

$$\begin{aligned} \phi(x_4) &= e^s[\phi(x_2), \phi(x_3)] \\ &= e^s[a_{12}x_1 + a_{22}x_2 + a_{32}x_3, a_{13}x_1 + a_{23}x_2 + a_{33}x_3] + v_4 \\ &= e^{s-t}(a_{22}a_{33} - a_{32}a_{23})x_4 + e^{s+t}(a_{12}a_{33} - a_{32}a_{13})x_5 \\ &\qquad + e^s(a_{12}a_{23} - a_{22}a_{13})x_6 + v_4, \end{aligned} \tag{12}$$

where v_4 is a vector in the center V_6.

The second relation $[x_1, x_3] = e^s x_5$ gives us

$$\begin{aligned} \phi(x_5) &= e^{-s}[\phi(x_1), \phi(x_3)] + v_5 \\ &= e^{-s}[a_{11}x_1, a_{13}x_1 + a_{23}x_2 + a_{33}x_3] + v_5 \\ &= e^{-s+t}a_{11}a_{33}x_5 + e^{-s}a_{11}a_{23}x_6 + v_5 \end{aligned} \tag{13}$$

for some vector $v_5 \in V_6$. Hence $\phi(x_5) \in V_4 \oplus V_6$. By Lemma 4.4, Part (1), the subspace $\mathbb{R}x_5$ is mapped into $\mathbb{R}x_5 \oplus V_3 \oplus V_6$ by ϕ. Hence, the

x_6 coefficient in Equation (13) is zero. From the fact that $a_{11} \neq 0$, we deduce that $a_{23} = 0$. Now we have

$$\phi(x_5) = e^{-s+t} a_{11} a_{33} x_5 + v_5. \tag{14}$$

Next we get

$$\begin{aligned} \phi(x_6) &= [\phi(x_1), \phi(x_2)] \\ &= [a_{11}x_1, a_{12}x_1 + a_{22}x_2 + a_{32}x_3] + v_6 \\ &= e^t a_{11}a_{32}x_5 + a_{11}a_{22}x_6 + v_6 \end{aligned} \tag{15}$$

for some vector $v_6 \in V_6$, so $\phi(x_6) \in V_4 \oplus V_6$.

The bracket relation $[x_2, x_6] = e^s x_7$ implies that

$$\begin{aligned} \phi(x_7) &= e^{-s}[\phi(x_2), \phi(x_6)] \\ &= e^{-s}[a_{12}x_1 + a_{22}x_2 + a_{32}x_3, e^t a_{11}a_{32}x_5 + a_{11}a_{22}x_6] \\ &= e^{-s+t} a_{11} a_{22}^2 x_7 + a_{11}(e^{-s-t} a_{12}a_{22} + e^{-s+2t} a_{32}^2) x_8. \end{aligned} \tag{16}$$

We use the relation $[x_3, x_4] = e^{-s} x_7$, substituting $a_{23} = 0$ when it occurs, to get

$$\begin{aligned} \phi(x_7) &= e^s[\phi(x_3), \phi(x_4)] \\ &= e^s[a_{13}x_1 + a_{23}x_2 + a_{33}x_3 + v_3, e^{s-t} a_{22}a_{33}x_4 \\ &\qquad + e^{s+t}(a_{12}a_{33} - a_{32}a_{13})x_5 - e^s a_{22}a_{13}x_6 + v_4] \\ &= e^{2s-2t} a_{33}^2 a_{22} x_7 \\ &\quad + \big(- e^{2s-t} a_{13}^2 a_{22} + e^{2s+2t} a_{33}(a_{12}a_{33} - a_{32}a_{13}) \big) x_8. \end{aligned} \tag{17}$$

The bracket relation $[x_1, x_6] = e^{-s} x_8$ gives

$$\begin{aligned} \phi(x_8) &= e^s[a_{11}x_1, e^t a_{11}a_{32}x_5 + a_{11}a_{22}x_6] \\ &= e^{s-t} a_{11}^2 a_{22} x_8, \end{aligned} \tag{18}$$

and from $[x_3, x_5] = e^s x_8$ we find

$$\begin{aligned} \phi(x_8) &= e^{-s}[a_{13}x_1 + a_{23}x_2 + a_{33}x_3, e^{-s+t} a_{11}a_{33}x_5] \\ &= e^{-2s+2t} a_{33}^2 a_{11} x_8. \end{aligned} \tag{19}$$

We know that $\phi(x_8) \neq 0$, so Equations (18) and (19) tell us that $a_{22} \neq 0$ and $a_{33} \neq 0$.

Equating coefficients of x_7 in Equations (16) and (17) gives

$$e^{-s+t} a_{11} a_{22} = e^{2s-2t} a_{33}^2, \tag{20}$$

and equating x_8 coefficients in Equations (18) and (19) gives

$$e^{s-t} a_{11} a_{22} = e^{-2s+2t} a_{33}^2. \tag{21}$$

Thus,

$$e^{3s-3t}a_{33}^2 = a_{11}a_{22} = e^{-3s+3t}a_{33}^2$$

from Equations (20) and (21). Because a_{33} is nonzero, we may conclude that $s = t$ as desired.

Now suppose that $m = 9$. The bracket relation $[x_2, x_3] = e^{4s}x_4$ implies that

$$\begin{aligned}\phi(x_4) = e^{4t-4s}&(a_{22}a_{33} - a_{32}a_{23})x_4 + e^{-4s-3t}(a_{12}a_{33} - a_{32}a_{13})x_5 \\ &+ e^{-4s-t}(a_{12}a_{23} - a_{22}a_{13})x_6 + v_4,\end{aligned} \tag{22}$$

where $v_4 \in V_6$. We use $[x_1, x_3] = e^{-3s}x_5$ to obtain that

$$\phi(x_5) = e^{3s-3t}a_{11}a_{33}x_5 + e^{3s-t}a_{11}a_{23}x_6 + v_5$$

for some $v_5 \in V_6$. By Lemma 4.4, we know that $\phi(x_5)$ is in the invariant subspace $\mathbb{R}x_5 \oplus \mathbb{R}x_5 \oplus V_4 \oplus V_6$, so the coefficient of x_6 is zero. Hence $a_{23} = 0$.

The bracket relation $[x_1, x_2] = e^{-s}x_6$ yields

$$\phi(x_6) = e^{s-3t}a_{11}a_{32}x_5 + e^{s-t}a_{11}a_{22}x_6 + v_6 \tag{23}$$

where $v_6 \in V_6$, and the bracket relation $[x_1, x_6] = e^{4s}x_8$ gives us

$$\phi(x_8) = e^{-3s+3t}a_{11}^2a_{22}x_8.$$

Because $\phi(x_8) \neq 0$, we see that $a_{22} \neq 0$.

The bracket relation $[x_2, x_6] = e^{-4s}x_7$ becomes

$$\begin{aligned}\phi(x_7) = e^{5s-5t}a_{11}a_{22}^2x_7 + a_{11}(e^{5s+3t}a_{22}a_{12} + e^{5s-7t}a_{32}^2)x_8 \\ + 2e^{5s-2t}a_{11}a_{22}a_{32}x_9.\end{aligned} \tag{24}$$

The bracket relation $[x_3, x_4] = e^{4s}x_7$ and that $a_{23} = 0$ gives

$$\begin{aligned}\phi(x_7) &= e^{-8s+8t}a_{22}a_{33}^2x_7 \\ &\quad + \left(-e^{-8s+3t}a_{22}a_{13}^2 + e^{-8s-7t}a_{33}(a_{12}a_{33} - a_{32}a_{13})\right)x_8 \\ &\quad - e^{-8s-2t}a_{22}a_{13}a_{33}x_9.\end{aligned} \tag{25}$$

Setting the x_7 coefficients from Equations (24) and (25) equal, we get

$$a_{11}a_{22} = e^{13t-13s}a_{33}^2. \tag{26}$$

The bracket relation $[x_3, x_5] = e^{-4s}x_8$ and the fact that $a_{23} = 0$ give

$$\phi(x_8) = e^{7s-7t}a_{33}^2a_{11}x_8.$$

This means that $a_{33} \neq 0$. Equating the x_8 coefficients in this expression and the previous expression for $\phi(x_8)$, we get

$$e^{-3s+3t}a_{11}^2a_{22} = e^{7s-7t}a_{33}^2a_{11},$$

so

$$a_{11}a_{22} = e^{10s-10t}a_{33}^2.$$

This together with Equation (26) gives $e^{13t-13s} = e^{10s-10t}$. Hence, $s = t$. $\square$

Theorem 4.6. *Let $m = 8$ or 9, and let $k \in \mathbb{N}$. Let $\mathfrak{n}_s^{m+2k}, s \in \mathbb{R}$ be the one-parameter family of nilpotent $(m+2k)$-dimensional nilpotent Lie algebras defined in Definition 4.1. None of the Lie algebras in the family are soliton.*

Proof. Let $x_{m+i} = y_i$ for $i = 1, \ldots, 2k$. Then the index set Λ with respect to the basis $\mathcal{B} = \{x_i\}_{i=1}^m \cup \{y_j\}_{j=1}^{2k} = \{x_i\}_{i=1}^{m+2k}$ is $\Lambda_{\mathfrak{m}} \cup \Lambda_{\mathfrak{h}}$, where $\Lambda_{\mathfrak{m}}$ is the index set for $\mathfrak{m}$ with respect to the basis $\{x_i\}_{i=1}^m$, and

$$\Lambda_{\mathfrak{h}} = \{(m+i, m+2k-i, 8) \,:\, i = 1, \ldots, k\}.$$

The set Λ is ordered as described in Section 2.2. In this ordering, if $(i_1, j_1, k_1) \in \Lambda_{\mathfrak{m}}$ and $(i_2, j_2, k_2) \in \Lambda_{\mathfrak{h}}$, then $(i_1, j_1, k_1) < (i_2, j_2, k_2)$.

First we consider the case that $m = 8$. Let the family $\mathfrak{n}_s^{8+2k}, s \in \mathbb{R}$ of nilpotent Lie algebras be as defined in Definition 4.1. We will do a proof by contradiction, so we suppose that $\mathfrak{n}_s^{8+2k}$ admits a soliton inner product.

Let U denote the Gram matrix for $\mathfrak{n}_s^{i+2k}$ with respect to the basis $\mathcal{B}$. For $a, b \in \mathbb{N}$, let $[0]_{a\times b}$ denote the $a \times b$ matrix with all entries zero, and let $[1]_{a\times b}$ denote the $a \times b$ matrix with all entries one, and let I_a denote the $a \times a$ identity matrix. The matrix U has block form

$$U = \begin{bmatrix} U_{11} & U_{12} & [0]_{5\times k} \\ U_{21} & U_{22} & [1]_{3\times k} \\ [0]_{k\times 5} & [1]_{k\times 3} & 3I_k \end{bmatrix},$$

where

$$U_8 = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}$$

is the matrix U in Equation (3), broken into blocks $U_{11}, U_{12}, U_{21}, U_{22}$ of sizes $5 \times 5, 5 \times 3, 3 \times 5$, and 3×3 respectively.

By Theorem 2.1 there exists a solution $\boldsymbol{v}$ to $U\boldsymbol{v} = [1]$ with all positive entries. We may write $\boldsymbol{v}$ as

$$\boldsymbol{v} = \begin{bmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \\ \boldsymbol{v}_3 \end{bmatrix},$$

where $\boldsymbol{v}_1 = (v_i)_{i=1}^5$ is 5×1, $\boldsymbol{v}_2 = (v_i)_{i=6}^8$ is 3×1 and $\boldsymbol{v}_3 = (v_i)_{i=9}^{8+k}$ is $k \times 1$. Multiplying $U\boldsymbol{v}$ in block form gives

$$\begin{aligned} U_{11}\boldsymbol{v}_1 + U_{12}\boldsymbol{v}_2 &= [1]_{5\times 1} \\ U_{21}\boldsymbol{v}_1 + U_{22}\boldsymbol{v}_2 + [1]_{3\times k}\boldsymbol{v}_3 &= [1]_{3\times 1} \\ [1]_{k\times 3}\boldsymbol{v}_2 + 3\boldsymbol{v}_3 &= [1]_{k\times 1}. \end{aligned}$$

Substituting $[1]_{3\times k}\boldsymbol{v}_3 = (\sum_{i=9}^{8+k} v_i)[1]_{3\times 1}$ and $[1]_{k\times 3}\boldsymbol{v}_2 = (\sum_{i=6}^{8} v_i)[1]_{k\times 1}$ into the above yields the equivalent system

$$U_{11}\boldsymbol{v}_1 + U_{12}\boldsymbol{v}_2 = [1]_{5\times 1} \tag{27}$$

$$U_{21}\boldsymbol{v}_1 + U_{22}\boldsymbol{v}_2 = \left(1 - \sum_{i=9}^{8+k} v_i\right)[1]_{3\times 1} \tag{28}$$

$$3\boldsymbol{v}_3 = \left(1 - \sum_{i=6}^{8} v_i\right)[1]_{k\times 1}. \tag{29}$$

Equation (29) implies that $\boldsymbol{v}_3 = c\,[1]_{k\times 1}$, where

$$c = \frac{1}{3}\left(1 - \sum_{i=6}^{8} v_i\right) > 0. \tag{30}$$

It follows that

$$\sum_{i=9}^{8+k} v_i = kc = \frac{k}{3}\left(1 - \sum_{i=6}^{8} v_i\right).$$

Now we bound the coefficient

$$a = 1 - \sum_{i=9}^{8+k} v_i$$

of $[1]_{3\times 1}$ in Equation (28). The matrices U_{21} and U_{22} are nontrivial and have no negative entries, and the entries of the vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are positive. Hence, Equation (28) forces a to be positive.

On the other hand,

$$\begin{aligned} a = 1 - \sum_{i=9}^{8+k} v_i &= 1 - kc \\ &= 1 - \frac{k}{3}\left(1 - \sum_{i=6}^{8} v_i\right) \\ &= \frac{3-k}{3} + \frac{k}{3}\sum_{i=6}^{8} v_i. \end{aligned}$$

The inequality in Equation (30) implies that $\sum_{i=6}^{8} v_i < 1$, so

$$a < \frac{3-k}{3} + \frac{k}{3} = 1.$$

Thus, we have shown that a lies in the interval $(0,1)$.

Returning to Equations (27)-(29), we have that

$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \end{bmatrix}$$

is a solution to the matrix equation

$$U_8\,\boldsymbol{x} = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} [1]_{5\times 1} \\ [a]_{k\times 1} \end{bmatrix}.$$

Now this involves only the matrix U_8, which is given explicitly in Equation (3). Solving symbolically using Matlab, we find that the general solution to the matrix equation is $\boldsymbol{v} = \boldsymbol{v}_0 + t\boldsymbol{v}_1$, where

$$\boldsymbol{v}_0 = \tfrac{1}{33}(7a+2, 3-6a, 13-4a, 11-11a, 2a+10, 13a-1, 11a-11, 0)^T$$

and

$$\boldsymbol{v}_1 = (-1, 1, 0, 1, -1, -1, 0, 1)^T.$$

For all such solutions $\boldsymbol{v} = (v_i)$, the component v_7 is $\frac{a-1}{3}$. We assumed that $v_7 > 0$, so then $a-1 > 0$, a contradiction to the fact that $a \in (0,1)$.

Therefore, for all $s \in \mathbb{R}$, and all $k \in \mathfrak{n}$, the Lie algebra $\mathfrak{n}_s^{8+2k}$ is not soliton, as claimed.

Now suppose that $m = 9$. Let $\mathfrak{n}_s^{9+2k}$ be one of the Lie algebras in the one-parameter family of nilpotent Lie algebras defined in Definition 4.1. Suppose that $\mathfrak{n}_s^{9+2k}$ admits a nilsoliton inner product.

Let U be the Gram matrix for $\mathfrak{n}_s^{9+2k}$ with respect to the basis $\{x_i\}_{i=1}^m \cup \{y_j\}_{j=1}^{2k} = \{x_i\}_{i=1}^{9+2k}$. By examining the index set Λ, we see that the matrix U has is of form

$$U = \begin{bmatrix} U_{11} & U_{12} & [0]_{8\times k} \\ U_{21} & U_{22} & [1]_{2\times k} \\ [0]_{8\times k} & [1]_{2\times k} & 3I_k \end{bmatrix},$$

where

$$U_9 = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}$$

is the 10×10 matrix U in Equation (4), and the blocks U_{11}, U_{12}, U_{21} and U_{22} are size $8\times 8, 8\times 2, 2\times 8$, and 2×2 respectively.

By Theorem 2.1, the matrix equation $U\boldsymbol{v} = [1]$ has a solution $\boldsymbol{v}$ with all positive entries. We write $\boldsymbol{v}$ as

$$\boldsymbol{v} = \begin{bmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \\ \boldsymbol{v}_3 \end{bmatrix},$$

where $\boldsymbol{v}_1$ is 8×1, $\boldsymbol{v}_2$ is 2×1 and $\boldsymbol{v}_3$ is $k \times 1$. Then

$$\begin{aligned} U_{11}\boldsymbol{v}_1 + U_{12}\boldsymbol{v}_2 &= [1]_{8\times 1} \\ U_{21}\boldsymbol{v}_1 + U_{22}\boldsymbol{v}_2 + [1]_{2\times k}\boldsymbol{v}_3 &= [1]_{2\times 1}. \end{aligned}$$

As $[1]_{2\times k}\boldsymbol{v}_3 = (\sum_{i=11}^{10+k} v_i)[1]_{2\times 1}$, we can rewrite this system as

$$U_{11}\boldsymbol{v}_1 + U_{12}\boldsymbol{v}_2 = [1]_{8\times 1} \tag{31}$$

$$U_{21}\boldsymbol{v}_1 + U_{22}\boldsymbol{v}_2 = \left(1 - \sum_{i=11}^{10+k} v_i\right)[1]_{2\times 1}. \tag{32}$$

Solving Equation (31) for $\left[\begin{smallmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \end{smallmatrix}\right]$, we get

$$\begin{bmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \end{bmatrix} = \boldsymbol{v}_0 + t_1\boldsymbol{w}_1 + t_2\boldsymbol{w}_2 + t_3\boldsymbol{w}_3,$$

where

$$\begin{aligned} \boldsymbol{v}_0 &= \tfrac{1}{11}(3, -1, 3, 0, 4, 4, 0, 0, 0, 0)^T \\ \boldsymbol{w}_1 &= (-1, 1, 0, 1, -1, -1, 0, 1, 0, 0)^T \\ \boldsymbol{w}_2 &= (-5, -2, 6, 0, -3, -3, 0, 0, 11, 0)^T \\ \boldsymbol{w}_3 &= (-5, 9, -5, 0, -3, -3, 0, 0, 0, 11)^T. \end{aligned}$$

But none of these solutions have all positive entries, a contradiction to Theorem 2.1. Therefore, for all $s \in \mathbb{R}$, the Lie algebra $\mathfrak{n}_s^{9+2k}$ does not admit a nilsoliton inner product. ☐

5. Proof of main theorem

Now we prove Theorem 1.4.

Proof. Let $\widetilde{\mathcal{N}}_n$ denote the moduli space of $\mathbb{N}$-graded, indecomposable nilpotent Lie algebras as described in Section 2.4. Let $\mathrm{Nonsol}(n) \subseteq \widetilde{\mathcal{N}}_n$ be the set of all $\overline{\mu}$ in $\widetilde{\mathcal{N}}_n$ such that $\mathfrak{n}_\mu$ does not admit a soliton inner product as described in Section 2.4.

We know from the results of Lauret, Will and Culma described in the introduction ([Lau02], [Wil03], [Cul11a], [Cul11b]) that the set of nonsoliton Lie algebras in $\widetilde{\mathcal{N}}_n$ is discrete when $n \leq 7$.

By Theorem 3.3, none of the 8-dimensional Lie algebras defined in Definition 3.1 are soliton. By Theorem 3.6 none of the 9-dimensional Lie algebras defined in Definition 3.4 are soliton. Theorem 4.6 implies that none of the Lie algebras in dimensions $n \geq 10$ defined in Definition 4.1 are soliton.

By Theorem 4.5, no two of the Lie algebras $\mathfrak{n}_s^n$ and $\mathfrak{n}_t^n$ defined in dimension $n \geq 8$ are isomorphic.

Therefore, in each dimension $n \geq 8$, the mapping $\gamma : \mathbb{R} \to \widetilde{\mathcal{N}}_n$ mapping $s \in \mathbb{R}$ to the equivalence class of the nilpotent Lie algebra $\mathfrak{n}_s^n$ is one-to-one, with image in Nonsol(n).

Hence, the set Nonsol(n) consisting of nonsoliton Lie algebras (modulo equivalence under isomorphism) in $\widetilde{\mathcal{N}}_n$ is not discrete when $n \geq 10$. $\square$

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Department of Mathematics, Idaho State University, 921 S. 8th Ave., Pocatello, ID 83209-8085

E-mail address: payntrac@isu.edu